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# $L_p$-dual three mixed quermassintegrals

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**Abstract**

In the paper, the concept of $L_p$-dual three-mixed quermassintegrals is introduced. The formula for the $L_p$-dual three-mixed quermassintegrals with respect to the $p$-radial addition is proved. Inequalities of $L_p$-Minkowski, and Brunn-Minkowski type for the $L_p$-dual three-mixed quermassintegrals are established. The new $L_p$-Minkowski inequality is obtained that generalize a family of Minkowski type inequalities. The $L_p$-Brunn-Minkowski inequality is used to obtain a series of Brunn-Minkowski type inequalities.

## 1 Introduction

The classical $L_p$-dual Minkowski inequality can be stated as follows (see [4]):

If $K$ and $L$ are star bodies and $0 < p \leq n$, then

$$\widetilde{V}_p(K,L)^n \leq V(K)^{n-p}V(L)^p, \tag{1.1}$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$.

Here, $V(K)$ denotes the ($n$-dimensional) Lebesgue measure of a body $K$ and call the volume of $K$. The $p$-dual mixed volume $\widetilde{V}_p(K,L)$, for $p \neq 0$, defined by

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L,u)^p dS(u), \tag{1.2}$$

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where, the letter $u$ for unit vectors, the surface of $B$ is $S^{n-1}$ and the letter $B$ is reserved for the unit ball centered at the origin, and $\rho(K,\cdot)$: $S^{n-1}\to[0,\infty)$, denotes the radial function of star body $K$, defined by (see e.g. [2] and [7])

$$\rho(K,u)=\max\{\lambda\geq 0:\lambda u\in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, $K$ will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. For any $p\neq 0$, the $p$-radial addition $K\widetilde{+}_pL$ defined by (see [3])

$$\rho(K\widetilde{+}_pL,u)^p=\rho(K,u)^p+\rho(L,u)^p, \tag{1.3}$$

for $K,L\in\mathcal{S}^n$. The Brunn-Minkowski inequality for the $p$-radial addition states that (see [3]): If $K,L\in\mathcal{S}^n$ and $0<p\leq n$, then

$$V(K\widetilde{+}_pL)^{p/n}\leq V(K)^{p/n}+V(L)^{p/n}, \tag{1.4}$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p<0$ or $p>n$. The operation of the $p$-radial addition and $L_p$-dual Minkowski, Brunn-Minkwski inequalities are the basic concept and inequalities in the $L_p$-dual Brunn-Minkowski theory.

In the paper, we give a generalization of the concept of the $p$-dual mixed volume. The $L_p$-dual three-mixed quermassintegrals is introduced. Let $K,L,Q\in\mathcal{S}^n, 0\leq i<n$ and $p\neq 0$, the $L_p$-dual three-mixed quermassintegrals of star bodies $K$, $L$ and $Q$, denoted by $\widetilde{W}_{p,i}(K,L,Q)$, defined by

$$\widetilde{W}_{p,i}(K,L,Q)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i-1-p}\rho(L,u)^p\rho(Q,u)dS(u). \tag{1.5}$$

When $Q=K$, the $L_p$-dual three-mixed quermassintegrals $\widetilde{W}_{p,i}(K,L,Q)=\widetilde{W}_{p,i}(K,L,K)$ becomes the $p$-dual mixed quermassintegrals $\widetilde{W}_{p,i}(K,L)$.

When $K=L$, the $L_p$-dual three-mixed quermassintegrals $\widetilde{W}_{p,i}(K,L,Q)=\widetilde{W}_{p,i}(K,K,Q)$ becomes the usual mixed quermassintegrals $\widetilde{W}_i(K,Q)$. When $K=L=Q$, the $L_p$-dual three-mixed quermassintegrals $\widetilde{W}_{p,i}(K,L,Q)=\widetilde{W}_{p,i}(K,K,K)$ becomes the usual dual quermassintegrals $\widetilde{W}_i(K)$.

The formula for the $L_p$-dual three-mixed quermassintegrals with respect to the $p$-radial addition is proved (see Section 3). The Minkowski inequality for the $L_p$-dual three-mixed quermassintegrals is obtained. If $K,L,Q\in\mathcal{S}^n$, $0\leq i<n$ and $0<p\leq n$, then

$$\widetilde{W}_{p,i}(K,L,Q)^{n-i}\leq\widetilde{W}_i(K)^{n-i-p-1}\widetilde{W}_i(L)^p\widetilde{W}_i(Q), \tag{1.6}$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p<0$ or $p>n$.

The new Minkowski inequality is obtained that generalize some Minkowski type inequalities. Taking $Q = K$ in (1.6), this becomes the following $L_p$-Minkowski inequality for $p$-dual quermassintegrals. If $K, L \in \mathcal{S}^n$, $0 < p \leq n$ and $0 \leq i < n$, then

$$\widetilde{W}_{p,i}(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n-i-p}\widetilde{W}_i(L)^p,$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$. Taking $K = L$ in (1.6), (1.6) becomes the following Minkowski inequality for dual quermassintegrals. If $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, then

$$\widetilde{W}_i(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n-i-1}\widetilde{W}_i(L),$$

with equality if and only if $K$ and $L$ are dilates. Taking $i = 0$ and $Q = K$ in (1.6), (1.6) becomes the following $L_p$-Minkowski inequsality. If $K, L \in \mathcal{S}^n$ and $0 < p \leq n$, then

$$\widetilde{V}_p(K, L)^n \leq V(K)^{n-p}V(L)^p,$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$. This is just the classical $L_p$-Minkowski inequality (1.1).

The Brunn-Minkowski inequality for the $L_p$-dual three-mixed quermassintegrals with respect to the $p$-radial addition is obtained. If $K, L, M, Q \in \mathcal{S}^n$, $0 \leq i < n-1$ and $0 < p \leq n$, then

$$\widetilde{W}_{p,i}(Q, K\widetilde{+}_pL, M) \leq \widetilde{W}_i(Q)^{(n-i-p-1)/(n-i)} \left(\widetilde{W}_i(K)^{p/(n-i)} + \widetilde{W}_i(L)^{p/(n-i)}\right)\widetilde{W}_i(M)^{1/(n-i)}, \quad (1.7)$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$.

The new Brunn-Minkowski inequality is used to obtain a family of Brunn-Minkowski type inequalities. Taking $Q = M = K\widetilde{+}_pL$ in (1.7), (1.7) becomes the following $L_p$-Brunn-Minkowski inequality for dual quermassintegrals. If $K, L \in \mathcal{S}^n$, $0 \leq i < n-1$ and $0 < p \leq n$, then

$$\widetilde{W}_i(K\widetilde{+}_pL)^{p/(n-i)} \leq \widetilde{W}_i(K)^{p/(n-i)} + \widetilde{W}_i(L)^{p/(n-i)},$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$. Taking $p = 1$ and $Q = M = K\widetilde{+}_pL$ in (1.7), (1.7) becomes the following Brunn-Minkowski inequality for dual quermassintegrals. If $K, L \in \mathcal{S}^n$ and $0 \leq i < n-1$, then

$$\widetilde{W}_i(K\widetilde{+}L)^{1/(n-i)} \leq \widetilde{W}_i(K)^{1/(n-i)} + \widetilde{W}_i(L)^{1/(n-i)},$$

with equality if and only if $K$ and $L$ are dilates.

Taking $i = 0$ and $Q = M = K\widetilde{+}_pL$ in (1.7), (1.7) becomes the following $L_p$-Brunn-Minkowdski inequality for volumes. If $K, L \in \mathcal{S}^n$ and $0 < p \leq n$, then

$$V(K\widetilde{+}_pL)^{p/n} \leq V(K)^{p/n} + V(L)^{p/n}, \tag{1.8}$$

with equality if and only if $K$ and $L$ are dilates.

The inequality is reversed for $p < 0$ or $p > n$. This is just classical $L_p$-Brunn-Minkowski type inequality (1.4).

## 2 Preliminaries

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. Associated with a compact subset $K$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K, \cdot) : S^{n-1} \to [0, \infty)$, defined by

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \mathcal{S}^n$, then (see e. g. [1])

$$\tilde{\delta}(K, L) = |\rho(K, u) - \rho(L, u)|_\infty.$$

### 2.1 Dual mixed volumes

The radial Minkowski linear combination, $\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r$, defined by (see [5])

$$\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r = \{\lambda_1 x_1 \widetilde{+} \cdots \widetilde{+} \lambda_r x_r : x_i \in K_i,\ i = 1, \ldots, r\},$$

for $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. It has the following important property:

$$\rho(\lambda K \widetilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot),$$

for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$.

If $K_i \in \mathcal{S}^n$ $(i = 1, 2, \ldots, r)$ and $\lambda_i$ $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the dual volume of $\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r$ is a homogeneous polynomial in the $\lambda_i$ given by (see [5])

$$V(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} \widetilde{V}_{i_1 \ldots i_n}, \tag{2.1}$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.1), it is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $\widetilde{V}(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} =$

$K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $\widetilde{V}(K_1 \ldots K_n)$ is written as $\widetilde{V}_i(K,L)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$, then the mixed volumes $V_i(K,B)$ is written as $\widetilde{W}_i(K)$ and is called the dual quermassintegral of star body $K$. Let $K_1 = \ldots = K_{n-i-1} = K$, $K_{n-i} = \ldots = K_{n-1} = B$ and $K_n = L$, then the dual mixed volume $\widetilde{V}(\underbrace{K,\ldots,K}_{n-1-i},\underbrace{B,\ldots,B}_{i},L)$ is written as $\widetilde{\widetilde{W}}_i(K,L)$ and is called the dual mixed quermassintegral of $K$ and $L$.

The dual quermassintegral of star body $K$, defined as an integral by (see [6]): If $K \in \mathcal{S}^n$ and $0 \le i < n$, then

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i} dS(u). \tag{2.2}$$

## 2.2 $p$-radial addition

For any $p \neq 0$, the $p$-radial addition $K \widetilde{+}_p L$ defined by (see [3])

$$\rho(K\widetilde{+}_p L, x)^p = \rho(K,x)^p + \rho(L,x)^p, \tag{2.3}$$

for $x \in \mathbb{R}^n$ and $K, L \in \mathcal{S}^n$. The following result follows immediately form (2.3).

$$\frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}_p\varepsilon\cdot L) - \widetilde{W}_i(L)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K.u)^{n-i-p}\rho(L.u)^p dS(u).$$

Let $K, L \in \mathcal{S}^n$, $p \neq 0$ and $0 \le i < n$, the $p$-dual mixed quermassintegral of star $K$ and $L$, $\widetilde{W}_{p,i}(K,L)$, defined by

$$\widetilde{W}_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i-p}\rho(L,u)^p dS(u). \tag{2.4}$$

Obviously, when $p = 1$, the $p$-dual mixed quermassintegral $\widetilde{W}_{p,i}(K,L)$ becomes the dual mixed quermassintegrals of star bodies $K$ and $L$ $\widetilde{W}_i(K,L)$. When $i = 0$, the $p$-dual mixed quermassintegral $\widetilde{W}_{p,i}(K,L)$ becomes the well-known $p$-dual mixed volume $\widetilde{V}_p(K,L)$.

This integral representation (2.4), together with the Hölder inequality, immediately gives that the following Minkowski inequality for $p$-dual quermassintegras: If $K, L \in \mathcal{S}^n$, $0 < p \le n$ and $0 \le i < n$, then

$$\widetilde{W}_{p,i}(K,L)^{n-i} \le \widetilde{W}_i(K)^{n-i-p}\widetilde{W}_i(L)^p, \tag{2.5}$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$.

It is easily seen that the $p$-dual mixed quermassintegral is linear with respect to the $p$-radial addition and together with inequality (2.5), show that the following Brunn-Minkowski inequality for $p$-radial addition: If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $0 < p \leq n$, then

$$\widetilde{W}_i(K \widetilde{+}_p L)^{p/(n-i)} \leq \widetilde{W}_i(K)^{p/(n-i)} + \widetilde{W}_i(L)^{p/(n-i)}, \tag{2.6}$$

with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$.

The operation of the $p$-radial addition and $L_p$-dual Minkowski, Brunn-Minkwski inequalities are the basic concept and inequalities in the $L_p$-dual Brunn-Minkowski theory. The latest information and important results of this theory can be referred to [8], [9], [10], [11] and [12] and the references therein.

## 3 $L_p$-dual three mixed quermassintegrals with respect to $p$-radial addition

In order to define the $L_p$-dual three-mixed quermassintegral with respect to $p$-radial addition, we need the following lemmas.

**Lemma 3.1** ([6]) *If $f_0, f_1$ and $f_2$ are (strictly) positive continuous functions defined on $S^{n-1}$ and $\alpha_1$, $\alpha_2$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u) f_1(u) f_2(u) dS(u) \leq \left( \int_{S^{n-1}} f_0(u) f_1^{\alpha_1}(u) dS(u) \right)^{1/\alpha_1} \left( \int_{S^{n-1}} f_0(u) f_2^{\alpha_2}(u) dS(u) \right)^{1/\alpha_2}, \tag{3.1}$$

*with equality if and only if there exist positive constants $\lambda_1$ and $\lambda_2$ such that $\lambda_1 f_1^{\alpha_1} = \lambda_2 f_2^{\alpha_2}$ for all $u \in S^{n-1}$.*

**Lemma 3.2** *Let $p \neq 0$, $0 \leq i < n$ and $\varepsilon > 0$. If $K, L \in \mathcal{S}^n$, then*

$$\lim_{\varepsilon \to 0^+} \frac{\rho(K \widetilde{+}_p \varepsilon \cdot L, u)^{n-i-1} - \rho(K, u)^{n-i-1}}{\varepsilon} = \frac{n-i-1}{p} \rho(K, u)^{n-i-p-1} \rho(L, u)^p. \tag{3.2}$$

*Proof* From (2.3) and in view of the L'Hôpital's rule, we obtain

$$\lim_{\varepsilon \to 0^+} \frac{\rho(K \widetilde{+}_p \varepsilon \cdot L, u)^{n-i-1} - \rho(K, u)^{n-i-1}}{\varepsilon}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0^+} \frac{(\rho(K,u)^p+\varepsilon\rho(L,u)^p)^{(n-i-1)/p}-\rho(K,u)^{n-i-1}}{\varepsilon}\\
&= \frac{n-i-1}{p}\lim_{\varepsilon\to 0^+}(\rho(K,u)^p+\varepsilon\rho(L,u)^p)^{(n-i-1-p)/p}\rho(L,u)^p\\
&= \frac{n-i-1}{p}\rho(K,u)^{n-i-1-p}\rho(L,u)^p.
\end{aligned}
$$

□

**Lemma 3.3** *Let $p\neq 0$, $0\le i<n$ and $\varepsilon>0$. If $K,L,Q\in\mathcal{S}^n$, then*

$$
\begin{aligned}
&\frac{p}{n-i-1}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}_p\varepsilon\cdot L,Q)-\widetilde{W}_i(K,Q)}{\varepsilon}\\
&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i-1-p}\rho(L,u)^p\rho(Q,u)dS(u). \quad (3.3)
\end{aligned}
$$

*Proof* This follows immediately from Lemma 3.2 and (2.2). □

**Definition 3.4** (The $L_p$-dual three-mixed quermassintegrals) Let $K,L\in\mathcal{S}^n, 0\le i<n$ and $p\neq 0$, the $L_p$-dual three-mixed quermassintegrals of star bodies $K$, $L$ and $Q$, denoted by $\widetilde{W}_{p,i}(K,L,Q)$, defined by

$$
\widetilde{W}_{p,i}(K,L,Q)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i-1-p}\rho(L,u)^p\rho(Q,u)dS(u). \tag{3.4}
$$

When $K=L=Q$, the $L_p$-dual three-mixed quermassintegrals $\widetilde{W}_{p,i}(K,L,Q)$ becomes the usual dual quermassintegrals $\widetilde{W}_i(K)$. When $p=1$, the $L_p$-dual three-mixed quermassintegrals $\widetilde{W}_{p,i}(K,L,Q)$ is written as $\widetilde{W}_i(K,L,Q)$ and call it dual three-mixed quermassintegrals of $K$, $L$ and $Q$. When $i=0$, the $L_p$-dual three-mixed quermassintegrals $\widetilde{W}_{p,i}(K,L,Q)$ becomes a new three-mixed volume, denoted by $\widetilde{V}_p(K,L,Q)$, and call it $L_p$-three dual mixed volume of $K$, $L$ and $Q$. When $p=1$, $\widetilde{V}_p(K,L,Q)$ becomes a new three-dual mixed volume, denoted by $\widetilde{V}(K,L,Q)$, and call it three dual mixed volume of $K$, $L$ and $Q$.

**Lemma 3.5** *If $K,L,Q\in\mathcal{S}^n$, $0\le i<n$ and $0<p\le n$, then*

$$
\widetilde{W}_{p,i}(K,L,Q)^{n-i-1}\le\widetilde{W}_i(K,Q)^{n-i-p-1}\widetilde{W}_i(L,Q)^p, \tag{3.5}
$$

*with equality if and only if $K$ and $L$ are dilates.*

*The inequality is reversed for $p<0$* or *$p<n$.*

*Proof* This follows immediately from (3.4) and Lemma 3.1. □

**Theorem 3.6** (The Minkowski inequality for $p$-dual three mixed quermassintegrals) *If $K,L,Q\in\mathcal{S}^n$, $0\le i<n$ and $0<p\le n$, then*

$$
\widetilde{W}_{p,i}(K,L,Q)^{n-i}\le\widetilde{W}_i(K)^{n-i-p-1}\widetilde{W}_i(L)^p\widetilde{W}_i(Q), \tag{3.6}
$$

*with equality if and only if $K$ and $L$ are dilates.*

*The inequality is reversed for $p < 0$ or $p > n$.*

*Proof* This follows immediately from Lemma 3.5 and inequality (2.5). $\square$

**Corollary 3.7** (The Minkowski inequality for dual three-mixed quermassintegrals) *If $K, L, Q \in \mathcal{S}^n$ and $0 \leq i < n$, then*

$$\widetilde{W}_i(K, L, Q)^{n-i} \leq \widetilde{W}_i(K)^{n-i-2}\widetilde{W}_i(L)\widetilde{W}_i(Q), \tag{3.7}$$

*with equality if and only if $K$ and $L$ are dilates.*

*Proof* This follows immediately from Theorem 3.6 with $p = 1$. $\square$

**Corollary 3.8** (The $L_p$-Minkowski inequality for $L_p$-dual three mixed volumes) *If $K, L, Q \in \mathcal{S}^n$, and $0 < p \leq n$, then*

$$\widetilde{V}_p(K, L, Q)^n \leq V(K)^{n-p-1}V(L)^p V(Q), \tag{3.8}$$

*with equality if and only if $K$ and $L$ are dilates. The inequality is reversed for $p < 0$ or $p > n$.*

*Proof* This follows immediately from Theorem 3.6 with $i = 0$. $\square$

**Corollary 3.9** (The Minkowski inequality for dual three mixed volumes) *If $K, L, Q \in \mathcal{S}^n$, then*

$$\widetilde{V}(K, L, Q)^n \leq V(K)^{n-2}V(L)V(Q), \tag{3.9}$$

*with equality if and only if $K$ and $L$ are dilates.*

*Proof* This follows immediately from Theorem 3.6 with $p = 1$ and $i = 0$.$\square$

**Corollary 3.10** *If $K, L, Q \in \mathcal{S}^n$, $0 \leq i < n$, then*

$$\widetilde{W}_{n,i}(K, L, Q)^{n-i}\widetilde{W}_i(K)^{i+1} \leq \widetilde{W}_i(L)^n\widetilde{W}_i(Q), \tag{3.10}$$

*with equality if and only if $K$ and $L$ are dilates.*

*Proof* This follows immediately from Theorem 3.6 with $p = n$. $\square$

**Theorem 3.11** (The $L_p$-Brunn-Minkowski inequality for the $p$-dual three mixed quermassintegrals) *If $K, L, M, Q \in \mathcal{S}^n$, $0 \leq i < n-1$ and $0 < p \leq n$, then*

$$\begin{aligned}&\widetilde{W}_{p,i}(Q, K\widetilde{+}_p L, M)\\ &\leq \widetilde{W}_i(Q)^{(n-i-p-1)/(n-i)}\left(\widetilde{W}_i(K)^{p/(n-i)} + \widetilde{W}_i(L)^{p/(n-i)}\right)\widetilde{W}_i(M)^{1/(n-i)},\end{aligned} \tag{3.11}$$

*with equality if and only if $K$ and $L$ are dilates.*

*The inequality is reversed for $p < 0$ or $p > n$.*

*Proof* From (2.3) and (3.4), it is easily seen that the $p$-dual three-mixed quermassintegral is linear with respect to the $p$-radial addition and together with inequality (3.6) show that for $0 < p \leq n$

$$\begin{aligned}
\widetilde{W}_{p,i}(Q, K\widetilde{+}_pL, M) &= \widetilde{W}_{p,i}(Q, K, M) + \widetilde{W}_{p,i}(Q, L, M) \\
&\leq \widetilde{W}_i(Q)^{(n-i-p-1)/(n-i)}\widetilde{W}_i(K)^{p/(n-i)}\widetilde{W}_i(M)^{1/(n-i)} \\
&+ \widetilde{W}_i(Q)^{(n-i-p-1)/(n-i)}\widetilde{W}_i(L)^{p/(n-i)}\widetilde{W}_i(M)^{1/(n-i)} \\
&= \left(\widetilde{W}_i(K)^{p/(n-i)} + \widetilde{W}_i(L)^{p/(n-i)}\right) \\
&\times \widetilde{W}_i(Q)^{(n-i-p-1)/(n-i)}\widetilde{W}_i(M)^{1/(n-i)}. \qquad (3.12)
\end{aligned}$$

From the equality condition of (3.6), the equality in (3.12) holds if and only if $K$ and $L$ are dilates of $Q$, this yields that the equality in (3.12) holds if and only if $K$ and $L$ are dilates. □

**Corollary 3.12** (The Brunn-Minkowski inequality for dual three-mixed quermassintegrals) *If* $K, L, M, Q \in \mathcal{S}^n$ *and* $0 \leq i < n-1$*, then*

$$\widetilde{W}_i(Q, K\widetilde{+}L, M) \leq \left(\widetilde{W}_i(K)^{1/(n-i)} + \widetilde{W}_i(L)^{1/(n-i)}\right)\widetilde{W}_i(Q)^{(n-i-2)/(n-i)}\widetilde{W}_i(M)^{1/(n-i)}, \quad (3.13)$$

*with equality if and only if* $K$ *and* $L$ *are dilates.*

*Proof* This follows immediately from Theorem 3.11 with $p = 1$. □

**Corollary 3.13** (The $L_p$-Brunn-Minkowski inequality for $p$-dual three mixed volumes) *If* $K, L, M, Q \in \mathcal{S}^n$ *and* $0 < p \leq n$*, then*

$$\widetilde{V}_p(Q, K\widetilde{+}_pL, M) \leq V(Q)^{(n-p-1)/n}\left(V(K)^{p/n} + V(L)^{p/n}\right)V(M)^{1/n}, \quad (3.14)$$

*with equality if and only if* $K$ *and* $L$ *are dilates.*

*The inequality is reversed for* $p < 0$ *or* $p > n$*.*

*Proof* This follows immediately from Theorem 3.11 with $i = 0$. □

**Corollary 3.14** (The Brunn-Minkowski inequality for dual three mixed volumes) *If* $K, L, M, Q \in \mathcal{S}^n$*, then*

$$\widetilde{V}(Q, K\widetilde{+}L, M) \leq V(Q)^{(n-2)/n}\left(V(K)^{1/n} + V(L)^{1/n}\right)V(M)^{1/n}, \quad (3.15)$$

*with equality if and only if* $K$ *and* $L$ *are dilates.*

*Proof* This follows immediately from Theorem 3.11 with $i = 0$ and $p = 1$. □

## 4 Conclusions

It is well known that the classical concept of mixed quermassintegrals of convex bodies generally refers to the mixing of two convex bodies. By means of variational technique, a new concept of mixed quermassintegrals of three convex bodies is proposed for the first time in $L_p$-space, which generalized the classical concept of two-mixed quermassintegrals of convex bodies. Further, the Minkowski inequality, and Brunn-Minkowki inequality for the three-mixed quermassintegrals are established, respectively. Therefore, a series of Minkowski type, and Brunn-Minkowski type inequalities are derived.

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